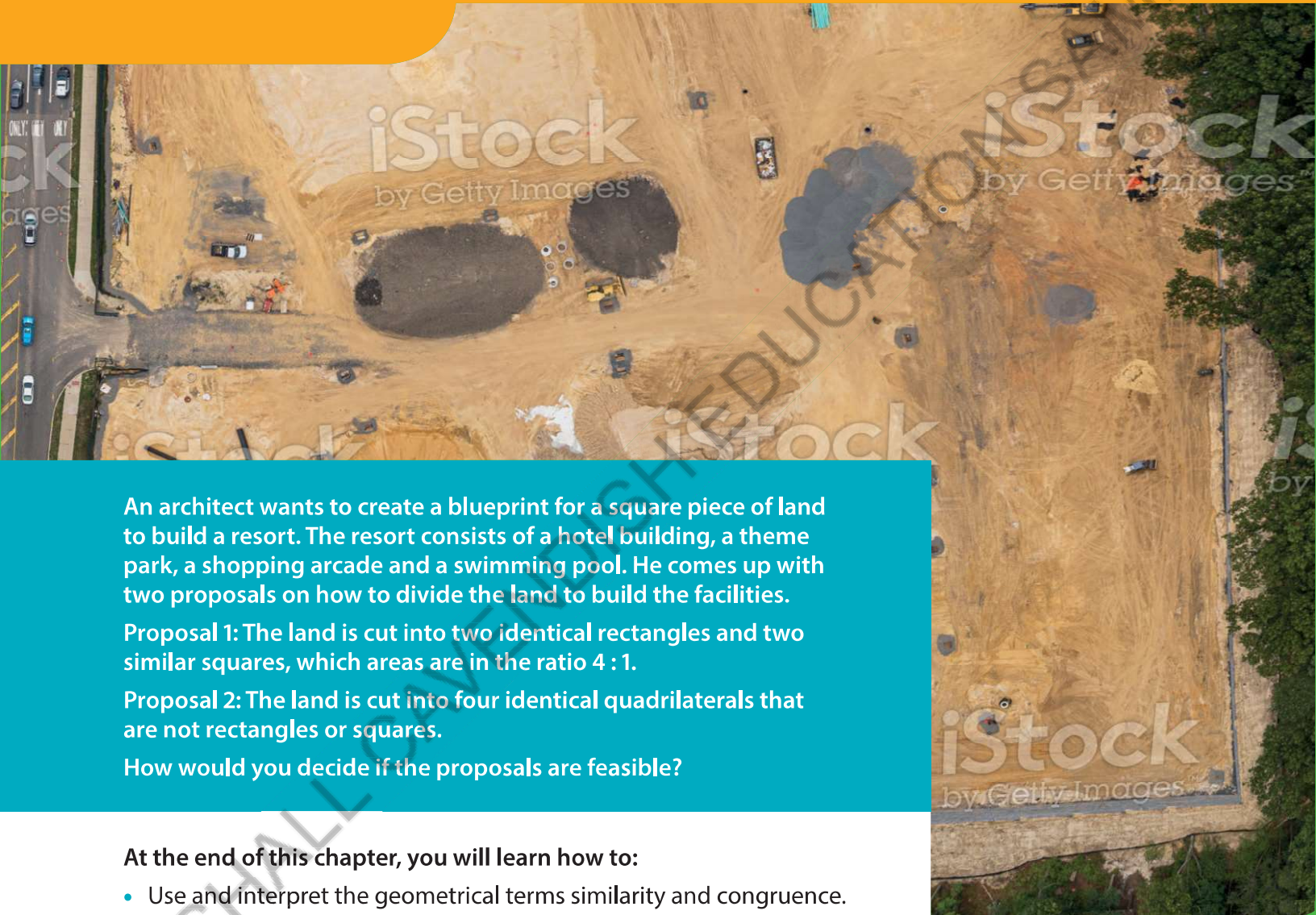# Congruence and Similarity

An architect wants to create a blueprint for a square piece of land to build a resort. The resort consists of a hotel building, a theme park, a shopping arcade and a swimming pool. He comes up with two proposals on how to divide the land to build the facilities.

Proposal 1: The land is cut into two identical rectangles and two similar squares, which areas are in the ratio 4 : 1.

Proposal 2: The land is cut into four identical quadrilaterals that are not rectangles or squares.

How would you decide if the proposals are feasible?

**At the end of this chapter, you will learn how to:**

- Use and interpret the geometrical terms similarity and congruence.
- Recognise congruent shapes.
- **S** Use the basic congruence criteria for triangles (SSS, ASA, SAS, RHS).
- Calculate lengths of similar figures.
- **S** Calculate lengths of similar figures.
  Use the relationships between areas of similar triangles, with corresponding results for similar figures and extension to volumes and surface areas of similar solids.

Recall

1 Draw the line of symmetry for the isosceles triangle ABC.
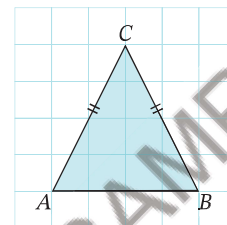
2 Triangle $BA'C'$ can be obtained by rotating triangle BAC by _____ degrees about B in a clockwise direction.
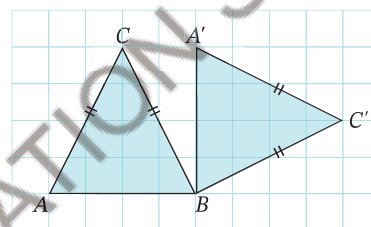
3 Use a ruler and a pair of compasses to construct a triangle with sides 3 cm, 4 cm and 5 cm.

4 If $\left(\frac{2}{3}\right)^2 = \frac{y}{27}$, find the value of y.

Only selected pages for Section 10.1 Congruent Figures are included in this submission.

10.1 Congruent Figures

In geometry, if two figures coincide or overlap exactly, they are identical. This means that the two figures have the same size (i.e. equal corresponding sides) and the same shape (i.e. equal corresponding angles). We say that the identical figures are **congruent**.

Checking the size and shape of the two figures is the first step to ascertain if two figures are congruent. Let us now explore whether some given figures are congruent.
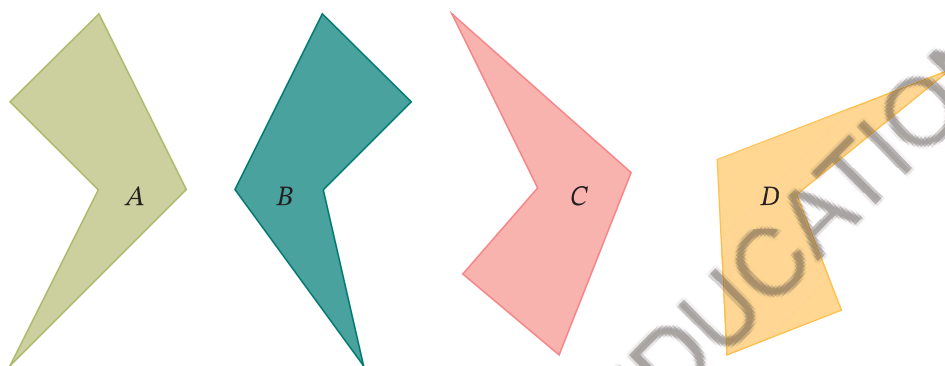
⏳ **Discover**

In this activity, you will learn to

- determine if a given set of figures are congruent.

Here are four pentagons.

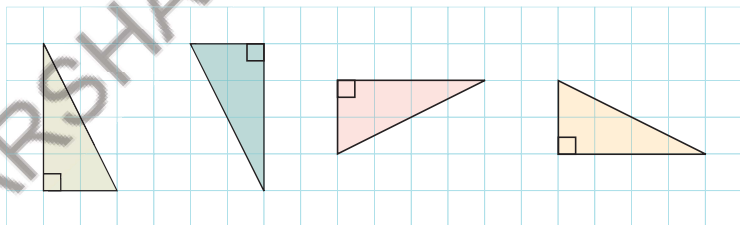How would you determine which figures are identical? Are they identical? Why?

⏳ **Think**   What do we mean by 'identical' in mathematics?

Key Results

Congruent figures have the same ⬚⬚⬚ (i.e. equal corresponding sides) and same ⬚⬚⬚ (i.e. equal corresponding angles). They are identical but they may have different ⬚⬚⬚.
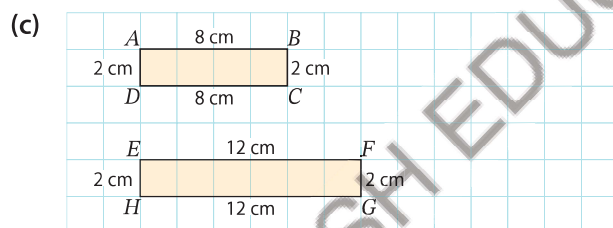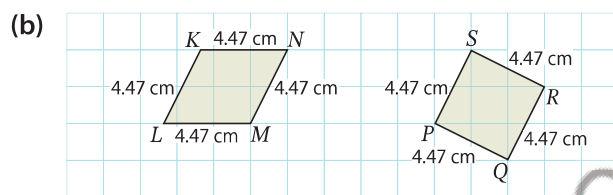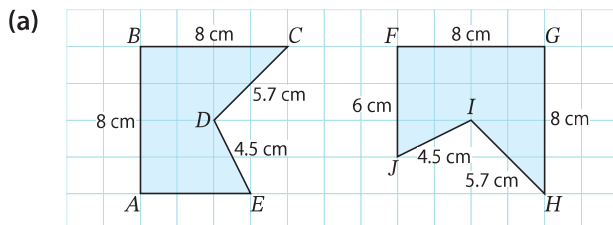
Here is an example.

All the four right-angled triangles are congruent, but their orientations are different.

Worked Example 1

(Determine if two figures are congruent)

Explain if the following pairs of figures are congruent.

(a)

(b)

(c)

(a) $AB = FG$ (equal corr. sides) $\qquad$ $\angle A = \angle F$ (equal corr. angles)

$BC = GH$ $\qquad\qquad\qquad\qquad$ $\angle B = \angle G$

$CD = HI$ $\qquad\qquad\qquad\qquad$ $\angle C = \angle H$

$DE = IJ$ $\qquad\qquad\qquad\qquad$ $\angle D = \angle I$

$AE = FJ$ $\qquad\qquad\qquad\qquad$ $\angle E = \angle J$

The two figures have the same size and shape, so they are congruent.

(b) In $PQRS$, $\angle PQR = \angle QRS = \angle RSP = \angle SPQ = 90°$.

$\therefore PQRS$ does not have any angles that corresponds to those in $KLMN$.

$\therefore PQRS$ does not have the same shape as $KLMN$ and so it is not congruent to $KLMN$.
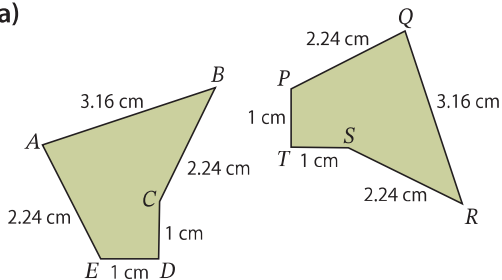
(c) In $EFGH$, $EF = HG = 12$ cm

$\therefore ABCD$ does not have the same lengths that corresponds to those in $EFGH$.

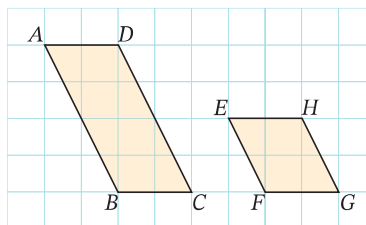$\therefore ABCD$ does not have the same size as $EFGH$ and so it is not congruent to $EFGH$.

PAUSE

Try 1

Explain if the following pairs of figures are congruent.

(a)

(b)

PAUSE

Discover

In this activity, you will learn to

- determine if a triangle and its translated image are congruent.

- determine if a triangle and its reflected image are congruent.

- determine if a triangle and its rotated image are congruent.

1

> Use a graphing software or grid paper to construct the two triangles T and T'.
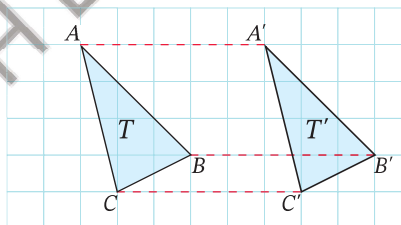
Points A, B and C are each translated horizontally 3 units to the right to form triangle T'. The images of A, B and C are A', B' and C' respectively.

(a) $AB = A'B'$, $BC = $ _____ and $AC = $ _____

$\angle A = $ _____ , $\angle B = $ _____ and $\angle C = $ _____

(b) Are triangles T and T' congruent? Why or why not?
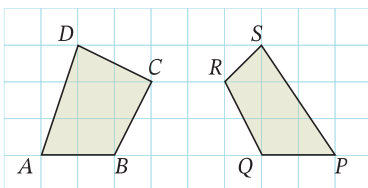
PAUSE

Think   Triangle T is transformed to T' by a translation. Will any triangle be congruent to its translated image?
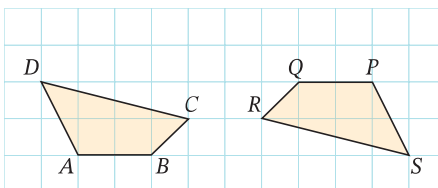
Exercise 10.1

1. Using the grids as reference, determine if each pair of figures are congruent. Explain your answers.
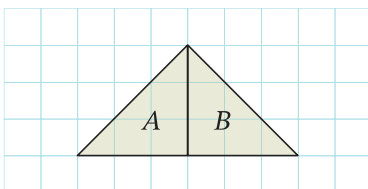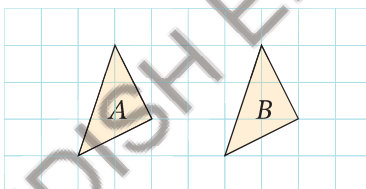
(a)

(b)

2. State if each of the following pairs of figures labelled as A and B are congruent. State a suitable transformation (rotation, reflection or translation) that give rise to the congruency.
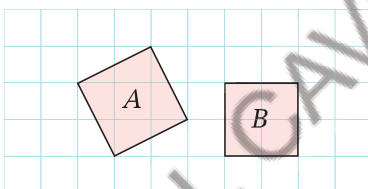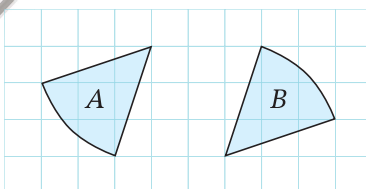
(a)

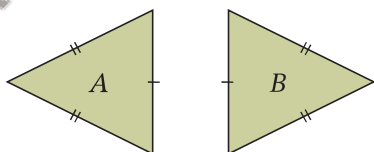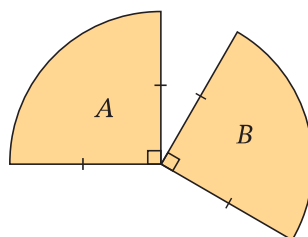**(b)**

(c)

(d)

3. Use transformations (reflection, rotation or translation) to determine if the figures marked with A and B are congruent. State the transformation.
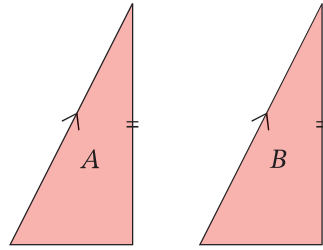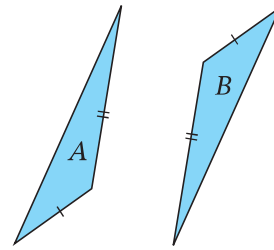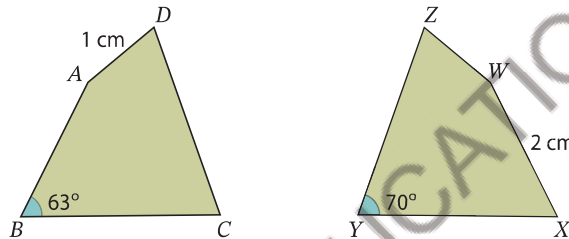
(a)

(b)

(c)

(d)

4. In the following diagram, the two quadrilaterals $ABCD$ and $WXYZ$ are congruent, that is $ABCD \equiv WXYZ$.

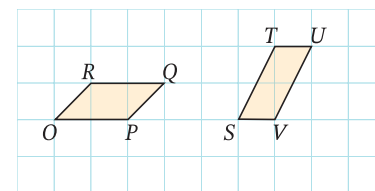Fill in the following blanks.

(a) $\angle ABC = \angle WXY = $

(b) $\boxed{} = \angle XYZ = $

(c) $AD = \boxed{} = \boxed{}$ cm

(d) $\boxed{} = WX = \boxed{}$ cm

> **TAKE NOTE**
>
> $ABCD \equiv WXYZ$
>
> You should identify the corresponding vertices first before filling up the blanks.
>
> $A \leftrightarrow W, B \leftrightarrow X,$
> $C \leftrightarrow Y, D \leftrightarrow Z$

5. Which of the following figures are congruent? Name them.

(a)

(b)

(c)

(d)

6. Triangles X and Y are congruent in each of the following figures. Write a congruency statement for the two triangles.

(a)

(b)

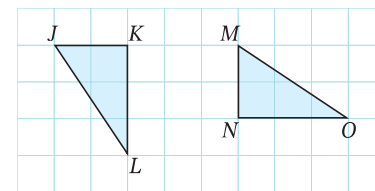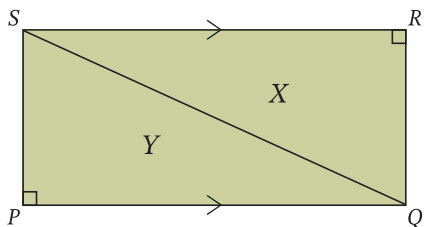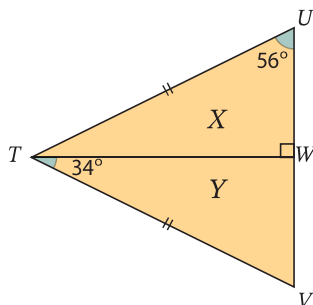7. $ABCDE$ is a regular pentagon. By using a transformation, explain why $\triangle AED \equiv \triangle BCD$.
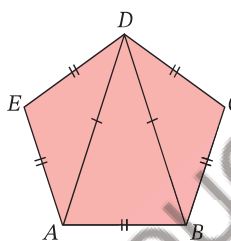
8. $\triangle ABC$ is congruent to $\triangle A'B'C'$. Given that $AB = 2$ cm, $BC = 3$, $\angle CAB = 56.3°$ and $A'C' = 3.6$ cm, find

   (a) the length of AC,

   (b) the length of $B'C'$ and

   (c) the angle $\angle B'C'A'$.
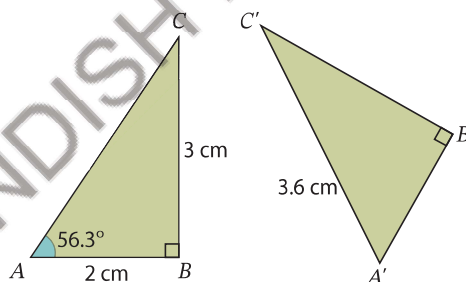
9. Trapezium $JKML$ is congruent to Trapezium $NMKO$.

   (a) By using a transformation, explain why Trapezium $JKML$ is congruent to Trapezium $NMKO$.

   (b) Identify the corresponding angle of $\angle KJL$ and the corresponding side of JK.
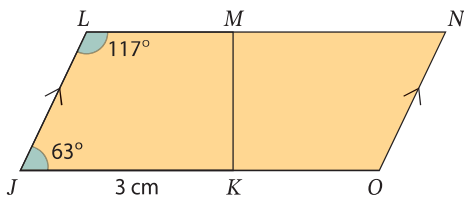
10.2 Congruency Tests

In the previous section, we learned that congruent figures have the same size (i.e. equal corresponding sides) and shape (i.e. equal corresponding angles). They can be mapped onto one another using reflection, rotation and translation.

Earlier, we have also learnt that a triangle has 6 measures; 3 sides and 3 angles. To construct a triangle, we only need to know:

- 3 sides,
- 2 sides and an included angle or
- 1 side and 2 angles.

Let us now explore how to compare two triangles to determine if they are congruent (having the same corresponding sides and angles). We will also explore the different congruency tests.

Discover

In this activity, you will learn to

- use the Side-Side-Side (SSS) Congruence Test for triangles.
- use the Angle-Side-Angle (ASA) Congruence Test for triangles.
- use the Side-Angle-Side (SAS) Congruence Test for triangles.
- use the Right-Hypotenuse-Side (RHS) Congruence Test for triangles.

1 (a) Can you construct two different triangles with sides 8 cm and 5 cm? Why or why not?

(b) Can you construct two different triangles with sides 8 cm, 5 cm and 6 cm? Why or why not?

Think Will your conclusion still be the same if you use different lengths?

(c) Compare the results in (a) and (b). What can you say?

2 (a) Draw a circle with radius 5 cm. Use the circle to construct two different triangles with sides 8 cm and 5 cm. Use the radius as one of the sides of the triangles. What condition needs to be included to make the triangles congruent?
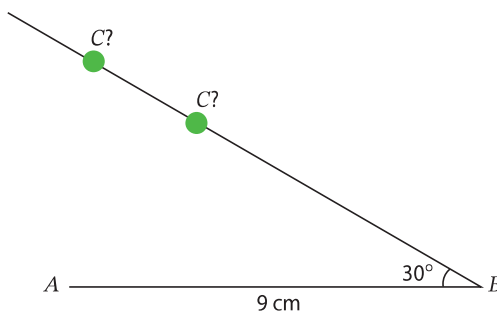
Think How does this condition ensure the congruency of the two triangles?

> **INTERESTING FACT**
>
> The triangle inequality theorem states that the sum of any two sides of a triangle is greater than or equal to the third side.

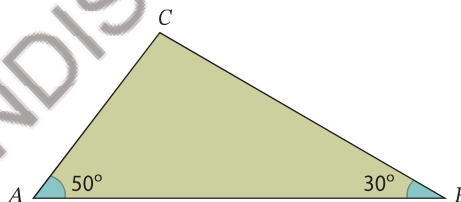**(b)** Construct $\triangle ABC$ with $AB = 9$ cm, $AC = 5$ cm and $\angle ABC = 30°$.

How many locations of C can you draw? What condition needs to be included to make this triangle unique?

PAUSE **Think** Compare the SAS and SSA conditions. What remains the same? What is different?

3(a) Construct a triangle that has angles $50°$ and $30°$. How many triangles can you draw? Are they unique?

(b) If the length of the side between the two angles, AB, is provided, can you construct two different triangles? Why or why not?

(c) If the length of the side AC is given instead, can you construct two different triangles? Why or why not?

PAUSE **Think** Does it matter if we write ASA Congruence Test or AAS Congruence Test? What is the difference?

4 Construct $\triangle ABC$ with $AB = 5$ cm, $AC = 7$ cm and $\angle ABC = 90°$. Construct as many such triangles as possible. Compare all the triangles you have constructed. Are they the same or different?

TAKE NOTE

In general, SSA is not a congruence test, but there are exceptions, such as the RHS Congruency Test

Key Results

1 **(i)** If the 3 _____ of a triangle are equal to the 3 corresponding _____ of another triangle, then the two triangles are _____.

   (ii) This means that all the corresponding _____ are equal.

   (iii) This is called the _____ Congruency Test.

2 **(i)** If 2 _____ and the included _____ of a triangle are equal to the 2 corresponding _____ and the corresponding included _____ of another triangle, then the two triangles are _____.

   (ii) This is called the _____ Congruency Test.

3 **(i)** If 2 _____ and 1 _____ of a triangle are equal to the 2 corresponding _____ and the corresponding _____ of another triangle, then the two triangles are _____.

   (ii) This is called the _____ Congruency Test.

4 **(i)** If the _____ and 1 _____ of a right-angled triangle are equal to the _____ and 1 _____ of another right-angled triangle, then the two triangles are _____.

   (ii) This is called the _____ Congruency Test.

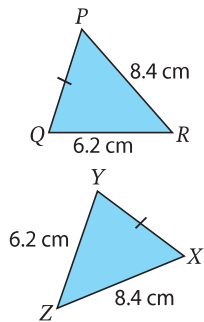**INTERESTING FACT**

Congruent triangles are used in the construction of bridges.

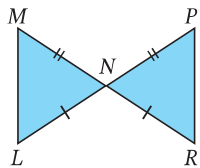Prove that $\triangle PQR$ is congruent $\triangle XYZ$.

TAKE NOTE ⓘ

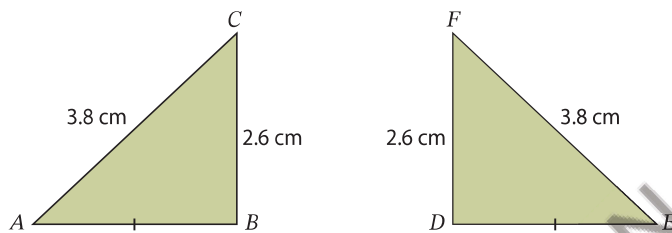The vertices of congruent triangles must be written in a corresponding order.

Prove that $\triangle LMN$ is congruent $\triangle PRN$.

Worked Example 6

(Prove two triangles are congruent using the SSS Congruency Test)

Prove that $\triangle ABC$ is congruent to $\triangle EDF$.
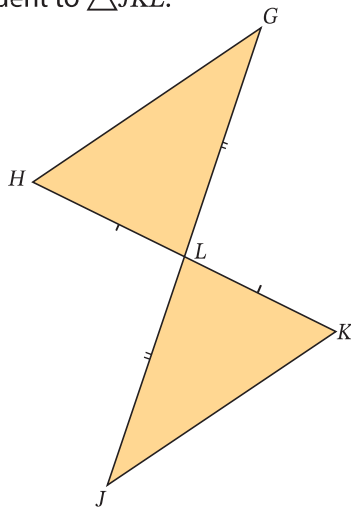
$AB = ED$ (given)

$AC = EF = 3.8$ cm

$CB = FD = 2.6$ cm

$\therefore \triangle ABC \equiv \triangle EDF$ (SSS)

Worked Example 7

(Prove two triangles are congruent using the SAS Congruency Test)

Prove that $\triangle GHL$ is congruent to $\triangle JKL$.

$HL = LK$ (given)

$GL = JL$ (given)

$\angle GLH = \angle JLK$ (vert. opp ∠s)

$\therefore \triangle GHL \equiv \triangle JKL$ (SAS)

● **Exercise 10.2**
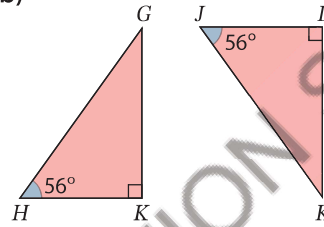
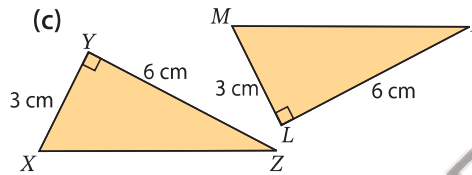1. Determine if the following pairs of triangles are congruent. If they are, state the reason for congruence.
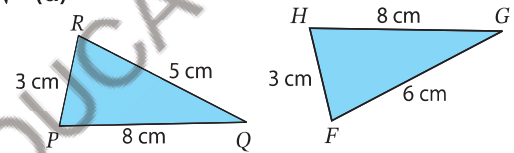
    (a)
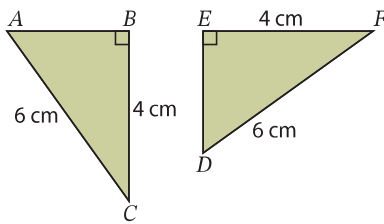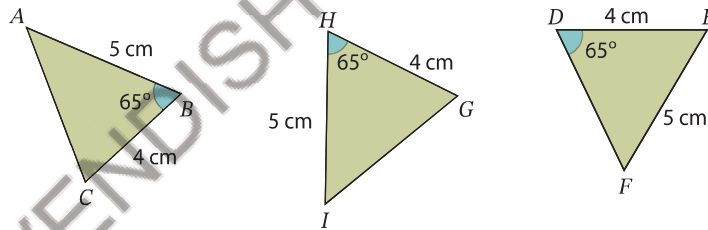

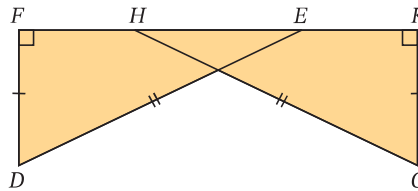   **(b)**

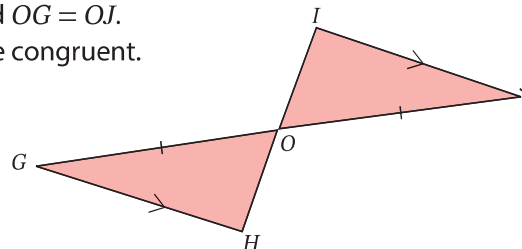
    (c)

    (d)

2. Determine which two triangles are congruent. State the reason for congruence.

3. In the diagram, $\triangle DFE$ and $\triangle GKH$ are two right-angled triangles. $DF = GK$ and $DE = GH$. Determine if $\triangle DFE$ is congruent to $\triangle GKH$. If yes, state the reason of congruence.
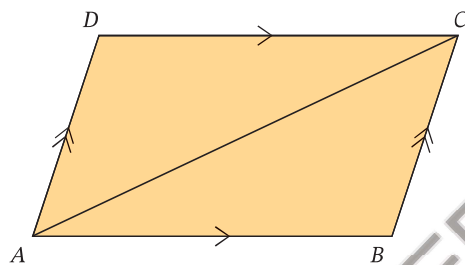

4. In the diagram, $GH // IJ$ and $OG = OJ$.
Prove that the two triangles are congruent.

5. In the diagram, AOB and POQ are straight lines. $OA = OB$ and $OP = OQ$.
   Prove that $AP \parallel QB$.
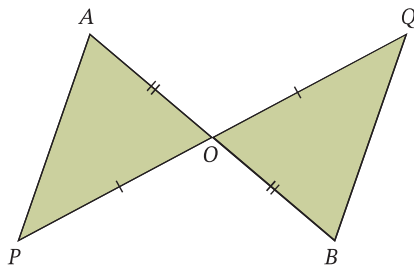
6. In the diagram, $ABCD$ is a parallelogram. Use a congruence test to explain why AC divides parallelogram $ABCD$ into two congruent triangles.

7. In the diagram, ABC is an equilateral triangle. $CD \parallel BA$ and $AE = DC$. Determine if $\triangle ABE$ is congruent to $\triangle CAD$. If yes, state the reason of congruence.
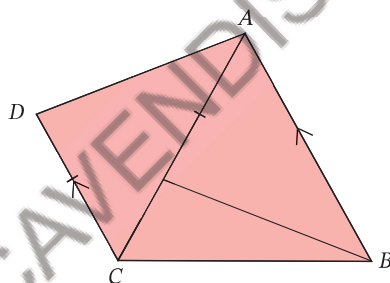
8. In the diagram, ABC is an isosceles triangle with $AB = BC$. P and Q lie on AC and AB respectively. BP bisects $\angle ABC$ and CQ bisects $\angle ACB$. Determine a pair of congruent triangles and state the reason of congruence.
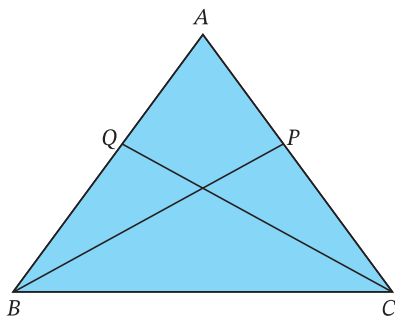
WB Further Exercise 10.2

Student Book pages for Section 10.3 Similar Figures, Section 10.4 Areas of Similar Figures and Section 10.5 Volumes of Similar Figures are not included in this submission.

Review

Put a tick if you are able to do the following tasks.

| Learning Outcomes	I can do the following:	✓
Use and interpret the geometrical terms similarity and congruence.	1. Describe the meaning of congruence.	☐
	2. Describe the meaning of similarity.	☐

Recognise congruent shapes.	1. Identify two congruent plane figures using:
	  i. Translation
	  ii. Reflection
	  iii. Rotation
		☐

S Use the basic congruence criteria for triangles (SSS, SAS, AAS, RHS).	1. Identify and name two congruent figures using the following congruence tests:
	  i. Side-Side-Side Congruence Test
	  ii. Side-Angle-Side Congruence Test
	  iii. Angle-Angle-Side Congruence Test
	  iv. Right-Hypotenuse-Side Congruence Test
		☐

Calculate lengths of similar figures.	1. Identify and name two similar figures using the following similarity tests:
	  i. Angle-Angle-Angle Similarity Test
	  ii. Side-Side-Side Similarity Test
	  iii. Side-Angle-Side Similarity Test
		☐

	2. Calculate lengths of similar figures.	☐

S Calculate lengths of similar figures. Use the relationships between areas of similar triangles, with corresponding results for similar figures and extension to volumes and surface areas of similar solids.	1. Use ratios of the corresponding sides between similar figures to calculate
	  i. Surface area
	  ii. Volume
		☐

Review Questions

1. In the diagrams, the pairs of figures are congruent. Write down the transformation that explains the congruency.
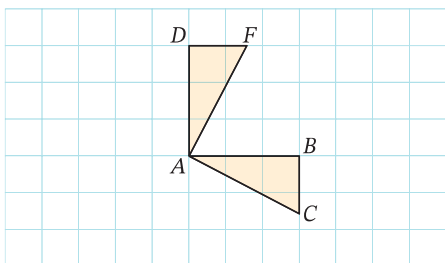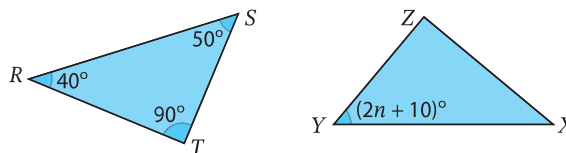
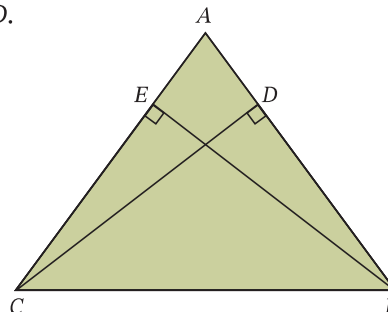   **(a)**
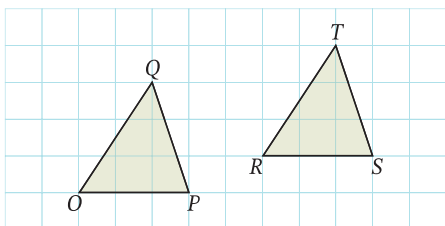
   (b)
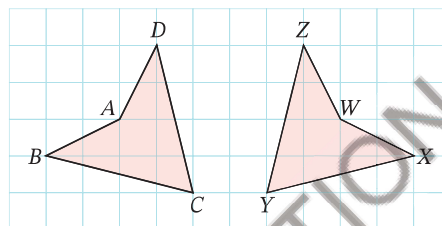
   (c)

2. **S** In $\triangle PQR$ and $\triangle XYZ$, $PQ = XY$, $QR = YZ$ and $\angle PQR = \angle XYZ$. Determine if $\triangle PQR$ is congruent to $\triangle XYZ$. If yes, state the reason of congruence.

3. $\triangle RST$ is congruent to $\triangle XYZ$. Find the value of n.
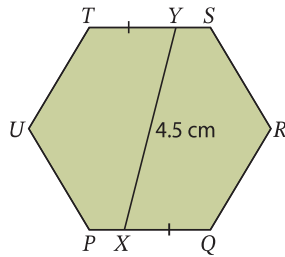
4. **S** In the diagram, BE and CD are heights of $\triangle ABC$. $BE = CD$.

   (a) Determine if $\triangle BAE$ is congruent to $\triangle CAD$.

   **(b)** What can you conclude about $\triangle ABC$?

5. **S** In the diagram, $PQRSTU$ is a regular hexagon with each side measuring 2.5 cm. $XQ = YT$ and $XY = 4.5$ cm.

    (a) Determine which triangle is congruent to $\triangle TYX$ and state the reason of congruence.

    (b) Find the perimeter of pentagon $TYXPU$.


6. In the diagram, F and G are points on EC and ED respectively. $\angle EFG = \angle CDE = 67°$. $DG = 4$ cm, $GE = 12$ cm, $FE = 8$ cm and $CF = x$ cm.

    (a) Determine which triangle is similar to $\triangle CED$ and give the reason of similarity.

    (b) Find the value of x.
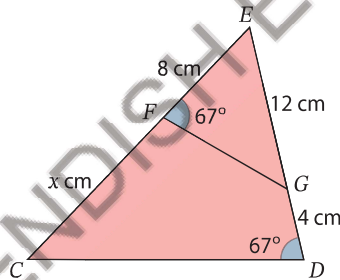
7. In the diagram, AY and BX intersects at C. $AB \text{ // } UV \text{ // } XY$. $AB = 10$ cm, $UV = 9$ cm, $CU = VY = 6$ cm, $XY = p$ cm, $BC = q$ cm and $UX = r$ cm. By identifying suitable similar triangles in the diagram, find the values of p, q and r, leaving your answers as fraction.
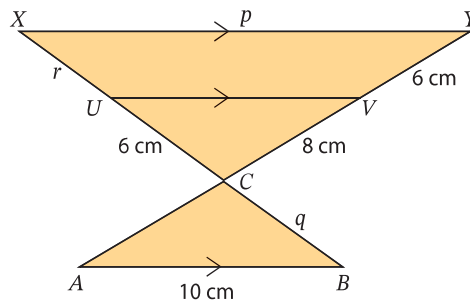
8. **S** In the diagram, D and E are points on AC and AB respectively. CB is parallel to DE. $AC = 10$ cm, $AE = 9$ cm and $EB = 6$ cm.

   (a) Determine which triangle is similar to $\triangle ADE$ and give the reason of similarity.
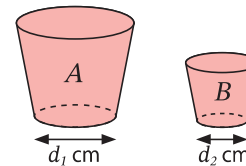
   (b) Find the length of DC.

   (c) Given that the area of $\triangle ABC$ is 25 cm², find the area of $\triangle ADE$.
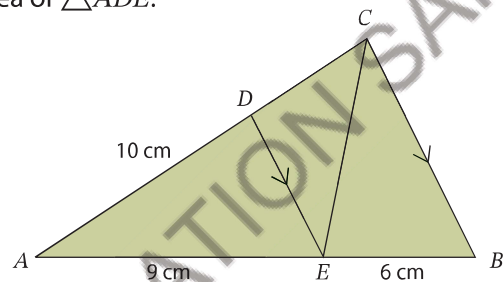
9. **S** The diagram shows two similar containers, A and B, with diameters d_1 cm and d_2 cm. Their base areas are 100 cm² and 64 cm² respectively.

   (a) Find $\frac{d_1}{d_2}$

   (b) Given that the volume of Container B is 200 cm³, find the volume of Container A.

10. **S** The diagram shows two similar cans, A and B. Their base areas are 16 cm² and 25 cm² respectively.

    (a) If the Can A can hold 400 cm³ of liquid, how much water can Can B hold?

    (b) Another can, Can C, is similar to Cans A and B. If Can C can hold 100 cm³ of liquid, does this mean that the height of Can C is half of Can A? Explain.
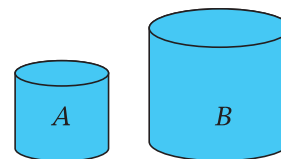
11. **S Paper Cup Design.** A student wants to design a paper cup in the shape of an inverted cone of height 6 cm and a surface area of 108 cm². When 96 cm³ of water is poured into the cup, the depth of water is 4 cm. By using the properties of two similar solids, find

    (a) the surface area of the cone which is wet.

    (b) the capacity of the paper cup.

    (c) the volume of water the paper cup still can hold.