CHAPTER

12 Binomial Theorem

When you save money in a bank, you are paid a yearly interest. Suppose the bank offers you an annual rate of 1% compound interest and you put \$500 into a saving account for a total of 10 years. The exact amount in your account after 10 years is $\$500 \times 1.01^{10}$. How would you get an approximate answer without the use of a calculator?

At the end of this chapter, you will learn how to:

- use the Binomial Theorem for expansion of $(a + b)^n$, for positive integer n
- use the general term $\binom{n}{r} a^{n-r}b^r$, $0 \le r \le n$

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Recall

1. Expand $a(b + c)$ and $(a + b)^2$.
2. Simplify $(a^m)^n$.
3. State $\binom{n}{r}$.
4. State $n!$.
5. State $0!$.

TAKE NOTE

$a^2 + 2ab + b^2$ has three **terms**: a^2, $2ab$ and b^2.

The **coefficients** of a^2 and b^2 are both 1 because $a^2 = 1a^2$ and $b^2 = 1b^2$.

The coefficient of ab is 2.

12.1 Binomial Expansion

Consider the identity $(a + b)^2 = a^2 + 2ab + b^2$.

The expression $(a + b)$ contains two terms, a and b, connected by a "+" sign.

An expression containing the sum of two terms is called a **binomial**.

PAUSE Think Is the expression $(a - b)$ a binomial? Why? What are two other examples of a binomial?

In this section, we will learn to expand the cube and higher powers of binomials, for example, $(a + b)^3$, $(a + b)^4$, $(a + b)^7$ and $(a + b)^{10}$.

When we expand $(a + b)^n$, where n is a non-negative integer, we are multiplying the products of n binomials into a sum of terms. This expansion is known as a **binomial expansion**.

Let us now explore how we can expand $(a + b)^3$, $(a + b)^4$ and $(a + b)^5$.

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Discover

In this activity, you will learn to

- expand $(a + b)^n$ for $n = 3, 4$ and 5.
- make five observations about the expansions.
- use the five observations to write the expansion of $(a + b)^n$.
- use Pascal's Triangle to expand $(a + b)^n$.

1 (a) Expand $(a + b)^3$.

$(a + b)^3 = (a + b)(a + b)^2$
$= (a + b)(a^2 + 2ab + b^2)$
$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$
$=$ ______
$= a^3 + 3a^2b +$ ___ $ab^2 +$ ___ b^3

(b) Expand $(a + b)^4$.

$(a + b)^4 = (a + b)(a + b)($ ___ $+$ ___ $)^2$.
$= (a + b)($ ______ $)$ **Hint:** Use the expansion of $(a + b)^3$ above.
$=$ ______
$=$ ______
$= a^4 + 4a^3b +$ ___ $a^2b^2 +$ ___ $ab^3 +$ ___ b^4

TAKE NOTE

Distributive Law

The expansion of $(a + b)Y$ is $aY + bY$.

If $Y = (a + 1)$, then
$(a + b)(a + 1)$
$= a(a + 1) + b(a + 1)$

If $Y = (a + c + 1)$, then
$(a + b)(a + c + 1)$
$= a(a + c + 1) +$
$b(a + c + 1)$

Think How can you expand $(a + b)^4$ using $(a + b)^2$ instead? Which method do you prefer? Why?

(c) Expand $(a + b)^5$. Show clearly how you did it.

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(d) Using the answers from **(a)** to **(c)**, complete the table below.

Binomial Expression	n	Expansion in descending powers of a	Number of terms in the expansion	Coefficient of the 2nd term
$(a + b)$	1	$a + b$	2	1
$(a + b)^2$	2	$a^2 + 2ab + b^2$	3	2
$(a + b)^3$	3			
$(a + b)^4$	4			
$(a + b)^5$	5			

INTERESTING FACT

Pascal's Triangle is a triangular array of numbers named after the French mathematician and philosopher, Blaise Pascal of the 17th century. Other mathematicians from India and China had studied the Pascal's Triangle centuries before Pascal.

There are many interesting patterns within the Pascal's Triangle. Here are some examples.

- Counting numbers (1, 2, 3, 4, 5, …)
- Triangular numbers (1, 3, 6, 10, 15, …)
- Perfect squares (4, 9, 16, …)

2 Examine the coefficient of the terms in the third column of the table above. The coefficients of the terms for each binomial expansion can be listed as follows:

n	Coefficients of the terms in the expansion $(a + b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

The array of numbers forms the **Pascal's Triangle**.

(a) What are the first and last numbers in each row of coefficients?

(b) How can you obtain the numbers in each row from the numbers in the previous row?

(c) How can you find the perfect squares in the Pascal's Triangle?

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Key Results

Look at the table in **(d)** on the previous page.
Make five observations about the expansion of $(a + b)^n$.

- There are ______ terms in the expansion.
- The power of a starts with n and decreases to ______.
  The power of b starts with ______ and increases to n.
- The coefficient of the 2nd term is always ______.
- The coefficients of the terms are ______.
- The sum of the powers of a and b in each term is always ______.

PAUSE **Think** What is the expansion of $(a + b)^n$ if $n = 0$?

In the Pascal's Triangle,

- each row starts and ends with ______.
- the remaining numbers in each row can be obtained from the previous row by ______ together the ______ numbers in the previous row which are closest to it.

The expansion of $(a + b)^n$, where n is a non-negative integer, can be generalised as:

$$c_0\, a^n + c_1\, a^{(n-1)}\, b + c_2\, a^{(n-2)}\, b^2 + \ldots + c_{(n-1)}\, ab^{(n-1)} + c_n\, b^n$$

where $c_0, c_1, c_2 \ldots, c_{(n-1)}, c_n$ are the coefficients of the respective terms.
The expansion is the sum of $(n + 1)$ terms, in ascending powers of b.

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You can now apply the observations in **Key Results** to expand $(a+b)^n$ for non-negative integer n.

Worked Example 1

(Expand binomial with sum of two variables using Pascal's Triangle)

(a) Using the Pascal's Triangle, write the coefficients of the terms in the expansion of $(a+b)^6$.

(b) Expand $(a+b)^6$.

Try 1

(a) Write down the line of coefficients of the terms in the expansion of $(a+b)^7$.
(b) Expand $(a+b)^7$.

(a) Using the coefficients of the terms for $(a+b)^5$: 1 5 10 10 5 1
The coefficients of the terms for $(a+b)^6$ are: 1 6 15 20 15 6 1

(b) Since the number of terms in the expansion of $(a+b)^n$ is $(n+1)$, there are seven terms in the expansion of $(a+b)^6$.

Step 1	The 1st term is $1(a^6)$.
Step 2	The coefficient of the 2nd term is 6. The variable component is a^5b because the power of a is 1 less than the 1st term and the power of b is 1 more. The sum of the powers of a and b is 6. So, the 2nd term is $6a^5b$.
Step 3	The coefficient of the 3rd term is 15. The variable component is a^4b^2. So, the 3rd term is $15a^4b^2$.
Step 4	The coefficient of the 4th term is 20. The variable component is $a^3\,b^3$. So, the 4th term is $20a^3\,b^3$.
Step 5	The 5th and 6th terms can be worked out in a similar manner. Since there are seven terms in this expansion, the 4th term is the middle term. By symmetry, the 5th term has the same coefficient as the 3rd term. Similarly, the 2nd and 6th terms have the same coefficient. Therefore, the 5th term is $15a^2b^4$, and the 6th term is $6ab^5$.
Step 6	The 7th term is $1(b^6)$.

Hence, putting all the terms together, we have
$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

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Worked Example 2

(Expand binomial with difference of two variables using Pascal's Triangle)

Expand $(a - b)^6$.

Rewrite $(a - b)^6$ as $(a + (-b))^6$.
From **Worked Example 1**,
$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Replacing b in the expansion with $-b$, we get
$(a + (-b))^6 = a^6 + 6a^5(-b) + 15a^4(-b)^2 + 20a^3(-b)^3 + 15a^2(-b)^4 + 6a(-b)^5 + (-b)^6$.

Therefore,
$(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

PAUSE Think

(a) Compare the two expansions of $(a + b)^6$ and $(a - b)^6$.
What remains the same? What is different?

(b) Can How can you expand $(a - b)^6$ without using the expansion of $(a - b)^6$?

TAKE NOTE

$(-y)^2$ is **not** $-y^2$.
It is
$(-y) \times (-y) = y^2$.
$(-y)^3 = (-y) \times (-y) \times (-y)$
$= -y^3$

PAUSE Try 2

Using the expansion of $(a + b)^7$ in the **Pause and Try 1**, expand $(a - b)^7$.

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Worked Example 3

(Expand binomial with sum of a variable and an integer using Pascal's Triangle)

Expand $(a + 2)^5$.

The coefficients of the six terms in the expansion of $(a + b)^5$ are $1,\ 5,\ 10,\ 10,\ 5,\ 1$.

Step 1	There are six terms in the expansion. The 1st term is $1(a^5)$.
Step 2	The variable component of the 2nd term is $(a^4)(2)$ because the power of a is 1 less than the 1st term and the power of 2 is 1 more. Its coefficient is 5. So, the 2nd term is $5(a^4)(2)$, which simplifies to $10a^4$.
Step 3	The variable component of the 3rd term is $(a^3)(2^2)$. Its coefficient is 10. So, the 3rd term is $10(a^3)(2^2) = 10(a^3)(4) = 40a^3$.
Step 4	The 4th and 5th terms can be worked out in a similar manner. The 4th term is $10(a^2)(2^3) = 80a^2$. The 5th term is $5(a)(2^4) = 80a$.
Step 5	The 6th term is $1(2^5) = 32$.

Hence, $(a + 2)^5 = a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$.

TAKE NOTE

The terms in the expansion of $(a + 2)^5$ are in **descending** powers of a.

PAUSE

Try 3

Expand $(a + 3)^4$.

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Worked Example 4

(Expand the product of a binomial and a higher power of another binomial using Pascal's Triangle)

(a) Write down, in ascending powers of k, the expansion of $(1 - 2k)^4$.

(b) Find, in ascending powers of k, the first three terms in the expansion of $(1 + k)(1 - 2k)^4$.

(a) The coefficients of the five terms in the expansion of $(a + b)^4$ are $1,\ 4,\ 6,\ 4,\ 1$.

So, $(1 - 2k)^4 = 1(1)^4 + 4(1)^3(-2k) + 6(1)^2(-2k)^2 + 4(1)(-2k)^3 + 1(-2k)^4$.

$$= 1(1) + 4(1)(-2k) + 6(1)(4k^2) + 4(1)(-8k^3) + 1(16k^4)$$
$$= 1 - 8k + 24k^2 - 32k^3 + 16k^4$$

(b) $(1 + k)(1 - 2k)^4 = (1+k)(1 - 8k + 24k^2 - 32k^3 + 16k^4)$

$$= (1)(1) + (1)(-8k) + (1)(24k^2) + (k)(1) + (k)(-8k) + \ldots$$
$$= 1 - 8k + 24k^2 + k - 8k^2 + \ldots$$
$$= 1 - 7k + 16k^2 + \ldots$$

PAUSE

Try 4

(a) Write down, in ascending powers of k, the expansion of $(1 + 3d)^4$.

(b) Find, in ascending powers of d, the first three terms in the expansion of $(2 - d)(1 + 3d)^4$.

TAKE NOTE

$(-2k)^2$ is **not** $-2k^2$. It is also **not** $-4k^2$.

It is $(-2k) \times (-2k) = 4k^2$.

$(-2k)^3 = (-2k) \times (-2k) \times (-2k) = -8k^3$

TAKE NOTE

The terms in the expansion of $(1 - 2k)^4$ are in **ascending** powers of k.

TAKE NOTE

In the expansion, the term with the lowest power of k is obtained by multiplying the constant terms because the power of k is 0. Therefore, the first three terms in the expansion of $(1 + k)(1 - 2k)^4$, in ascending powers of k, are up to k^2.

To obtain the constant term and the terms in k and k^2, we multiply 1 in $(1 + k)$ with the first three terms in the expansion of $(1 - 2k)^4$ and multiply k with the first two terms.

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PAUSE Try 5

Using your answers in **Pause and Try 4**, find the coefficient of d^3 in the expansion of $(1 + d)(1 + 3d)^4$.

TAKE NOTE

To determine the term in k^3 in the expansion of the product, we multiply 1 with $-32k^3$ and $2k$ with $24k^2$.

Worked Example 5

(Find the coefficient of a term in the expansion of the product of a binomial and a higher power of another binomial)

Using the answers in **Worked Example 4**, find the coefficient of k^3 in the expansion of $(1 + 2k)(1 - 2k)^4$.

$(1 + 2k)(1 - 2k)^4 = (1 + 2k)(1 - 8k + 24k^2 - 32k^3 + 16k^4)$

Term in $k^3 = (1)(-32k^3) + (2k)(24k^2)$

$= -32k^3 + 48k^3$

$= 16k^3$

Hence, the coefficient of k^3 is 16.

PAUSE Try 6

Find, in ascending powers of x, the first three terms in the expansion of $(1 + x)^6$. Then use the expansion to find the value of $(1.02)^6$.

TAKE NOTE

The coefficients of the first three terms in the expansion of $(a + b)^7$ are 1, 7, 21, ...

TAKE NOTE

Comparing $(0.98)^7$ with $(1 - 2x)^7$, equate $1 - 2x$ with 0.98, then solve for x.

That is, $1 - 2x = 0.98$

$2x = 0.02$

$x = 0.01$

Worked Example 6

(Estimate powers of a real number)

(a) Find, in ascending powers of x, the first three terms in the expansion of $(1 - 2x)^7$. Then use the expansion to find the value of $(0.98)^7$.

(b) Find the value of $(0.98)^7$ using a calculator.

(c) Compare the values of $(0.98)^7$ obtained in **(a)** and **(b)**. To how many decimal places is the approximated value in **(a)** corrected to?

(a) $(1 - 2x)^7 = (1)^7 + 7(1)^6(-2x) + 21(1)^5(-2x)^2 + \ldots$

$= 1 - 14x + 84x^2 + \ldots$

To estimate $(0.98)^7$ using $(1 - 2x)^7$, choose $x = 0.01$.

Substituting $x = 0.011$ into the first three terms in the expansion of $(1 - 2x)^7$,

$(0.98)^7 = 1 - 14(0.01) + 84(0.01)^2 + \ldots$

$= 0.8684$

(b) Using a calculator, $(0.98)^7 = 0.868\,125\,533$

(c) The approximated value is corrected to 3 decimal places.

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Exercise 12.1

Basic Level

1. Expand each of the following.

   (a) $(c + 2)^4$ **(b)** $(1 - h)^6$ **(c)** $(3 + p)^5$ **(d)** $(m - \frac{1}{2})^3$

2. Find, in descending powers of x, the expansion of each of the following.

   (a) $(2x + 3)^5$ **(b)** $(3x^2 - 1)^4$ **(c)** $(\frac{x}{4} - 2)^6$ **(d)** $(3x + \frac{1}{2})^5$

3. Find, in ascending powers of y, the expansion of each of the following.

   (a) $(3 - y)^5$ **(b)** $(1 + 2y)^6$ **(c)** $(2 + \frac{y}{3})^4$ **(d)** $(1 - y^2)^3$

Intermediate Level

4. The row of coefficients in the expansion of $(a + b)^n$ is

   $1, a, 55, b, 330, c, 462, d, 165, e, f, 1$

   Find the values of a, b, c, d, e, f and n. Explain clearly how you obtained each value.

5. Expand each of the following.

   (a) $(2a + 3b)^4$ **(b)** $(c - \frac{1}{c})^5$

6. Find the first four terms in the expansion of $(2 + 3y)^5$.

   Hence, deduce the first four terms in the expansion of each of the following.

   (a) $(2 - 3y)^5$

   (b) $(2 + 3y^2)^5$

7. **(a)** Expand $(2 - p)^6$.

   (b) Find, in ascending powers of p, the first three terms in the expansion of $(2 + p)(2 - p)^6$.

   (c) Find the coefficient of p^3 in the expansion of $(1 - \frac{2}{p})(2 - p)^6$.

8. Find the first four terms of the expansion of $(1 - 3x)^6$ in ascending powers of x.

   (a) If the expansion in **(a)** were to be used to approximate the value of $(0.97)^6$, what would be a suitable value of x to take?

   (b) By substituting the value of x obtained in **(b)** into the expansion in **(a)**, find an approximate value of $(0.97)^6$.

WB Further Exercise 12.1

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Student's Book pages for Section 12.2 on Expansion of $(a + b)^n$ using Binomial Theorem are not included in this submission.

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Review

Put a tick if you are able to do the following tasks.

Learning Outcomes	I can do the following:	✓
Use Pascal's Triangle for expansion of $(a+b)^n$, for positive integer n.	1. Write out the coefficients of the terms in the expansion of $(a+b)^n$, for n up to 10, using Pascal's Triangle.	☐
	2. Expand $(a+b)^n$ using Pascal's Triangle.	☐
Use the Binomial Theorem for expansion of $(a+b)^n$, for positive integer n.	1. State $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ for any positive integer n.	☐
	2. State $\binom{n}{r} = \frac{n}{n!(n-r)!}$, where n is a non-negative integer and r is an integer such that $0 \le r \le n$.	☐
	3. Recognise five observations about the expansion of $(a+b)^n$: (i) There are $n+1$ terms in the expansion. (ii) The power of a starts with n and decreases to 0. The power of b starts with 0 and increases to n. (iii) The coefficient of the 2nd term is always n. (iv) The coefficients of the terms are symmetrical. (v) The sum of the powers of a and b in each term is always n.	☐
	4. Apply the following: $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$ where n is a non-negative integer and r is an integer such that $0 \le r \le n$.	☐
Find the general term in the expansion of $(a+b)^n$.	Apply the $(r+1)$th term, denoted by $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.	☐

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Review Questions

1. After John had expanded $(a + 1)^6$ into a sum of terms, he made three observations. State whether each of his observations is true or false. If his observation is false, provide the correct answer.
   (a) There are 6 terms in the sum.
   (b) The coefficients of the terms are $1,\ 6,\ 15,\ 15,\ 6,\ 1$.
   (c) $6a$ and $6a^5$ are two terms in the sum.

2. Susan made the following two observations about the expansion of $(c - 2)^9$. State whether each of her observations is true or false. If her observation is false, provide the correct answer.
   (a) The 2nd term in the expansion is $-9c$.
   (b) The 5th term in the expansion is $\binom{9}{5}c^4(-2)^5$.

3. The row of coefficients in the expansion of $(a + b)^7$ is $1, 7, 21, 35, 35, 21, 7, 1$. Using Pascal's Triangle, write down the row of coefficients in the expansion of each of the following.
   (a) $(a + b)^8$
   (b) $(a + b)^9$

4. The expansion of $(2 + 3x)^6$ using the Binomial Theorem is shown below.
   Fill in the boxes and then simplify the terms.

$$(2 + 3x)^6 = 1(2)^6 + \square(2)^5(3x) + 15(2)^{\square}(3x)^2 + \square(2)^3(3x)^{\square}$$
$$+ 15(2)^{\square}(3x)^{\square} + \square(2)^{\square}(3x)^{\square} + 1(3x)^6$$

5. **(a)** Using the formula $\binom{n}{r} = \frac{n}{r!(n-r)!}$, evaluate each of the following.
   (i) $\binom{n}{0}$ **(ii)** $\binom{n}{n}$
   (b) Hence, write down a relationship connecting $\binom{n}{0}$ and $\binom{n}{n}$.
   (c) Draw the Pascal's Triangle for $n = 0, 1, 2, 3, 4$ and 5, circle the locations of $\binom{n}{0}$ and $\binom{n}{n}$ as n goes from 0 to 5.

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6. **(a)** Find, in ascending powers of u, the first three terms in the expansion of $(3 - u)^8$.
   (b) Use the expansion to estimate the value of $(2.98)^8$.
   (c) Find, to three decimal places, the value of $(2.98)^8$ using a calculator.
   (d) To how many decimal places is the estimated value in (b) corrected to when compared with the value in (c)?

7. **(a)** Find, in descending powers of t, the first four terms in the expansion of $(t^2 + 2)^7$.
   (b) Find the term in t^{12} in the expansion of $(1 - \frac{1}{t^2})(t^2 + 2)^7$.
   (c) Find the coefficient of t^8 in the expansion of $(1 - \frac{1}{t^2})(t^2 + 2)^7$.

8. The first four terms in the expansion of $(2x - \frac{1}{4})^9$ are $512x^9 - 576x^8 + px^7 + qx^6 + \ldots$.
   (a) Find the values of p and q.
   (b) Calculate the coefficient of x^6 in the expansion of $(4x + 1)(2x - \frac{1}{4})^9$.

9. The expansion of $(3 + kx)^9$, where k is a non-zero integer, is a polynomial in x. The coefficients of x^3 and x^4 are equal. Find the value of k.

10. **Finance.** David put \$500 in a savings account that offers 5% compound interest per annum for 10 years. The exact amount in David's account at the end of 10 years is $\$500 \times 1.05^{10}$.
   (a) Using the first two terms of the expansion of $(1 + x)^{10}$, approximate the value of 1.05^{10}.
   (b) Find an approximation to the exact amount in David's account at the end of 10 years.
   (c) It is given that the Error in the Amount = Actual Amount − Approximate Amount. Calculate the error in the amount when the approximated amount is used.
   (d) The relative error is calculated by taking the error divided by the actual amount. Calculate the relative error in the amount, correct to one decimal place when the approximated amount is used.

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