Chapter 12: Binomial Theorem

Content Matrix

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12.2 Expansion of $(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{n}}$ using Binomial Theorem	Pause and Discover, pp. 12–13 Key Results, p. 14 **Worked Example 7** and **Pause and Try 7**, p. 14 **Worked Example 8** and **Pause and Try 8**, p. 15 **Worked Example 9** and **Pause and Try 9**, p. 16 Exercise 12.2, p. 17	Further Exercise 12.2, pp. 5–8
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<u>LESSON 1</u>
Warm-up

Chapter Opener (p. 1)

Before getting students to read the chapter opener, have the class recap the idea of compound interest by illustrating how to compute the amount of savings in a bank account after 2 years using the information given in the chapter opener.

Then have students work out the amount in the account after 3 years. Encourage peer checking of solution before the teacher reveals the answer. Then ask students to predict the amount of savings in the account after 5 years. Invite one or two students to explain how they arrived at their answers.

Ask: How did you arrive at your estimation? What is a limitation of your method?
[Possible Answer: I got my estimated amount by taking 1% of $500 and adding the answer to the principal amount at the end of every year. A limitation is that it is a very tedious process.]

Briefly mention that the estimation of the total amount received in the savings plan is related to the Pascal's Triangle and the Binomial Theorem that they will be learning in this chapter.

Note: The Chapter Opener will be revisited in the Workbook under Chapter Opener Revisit.

Learning Objectives

State the learning objectives of this chapter:

- use the Binomial Theorem for expansion of $(a+b)^n$ for positive integer n
- use the general term $\binom{n}{r}a^{n-r}b^r$, $0 \leqslant r \leqslant n$

Recall **(p. 2)**

Ask students to attempt the diagnostic questions on the opening page of the chapter.
[Answers at MCEduHub]

Main Lesson Content

12.1 Binomial Expansion (pp. 2–11)

Introduce commonly used vocabulary such as 'binomial', 'term' and 'coefficient' using the identity $(a+b)^2 = a^2 + 2ab + b^2$.

Get students to consider the identity $(a+b)^2 = a^2 + 2ab + b^2$.
Explain that the expression $(a+b)$ contains two terms, *a* and *b*, connected by a + sign.
Point out that such an expression which contains the sum of two terms is called a **binomial.**

Take Note
Guide students to recall the terms and coefficients in $a^2 + 2ab + b^2$.

Pause and Think
Prompt students to explain why $(a-b)$ is a binomial. Point out that it is a binomial because it can be expressed as adding *a* and – *b*, that is, $(a + (-b))$. Other examples of binomial include $(a+3b)$, $(x-2)$, $(5+2k)$ and $(w^2 - yz)$.

Explain to students that they will learn to expand higher powers of binomials.
Show them some examples of binomial, term and coefficient.

Pause and Discover (pp. 3–4)

1(a) Encourage students to make a conjecture on the expansion of $(a+b)^3$ before going through the steps outlined in the Student's Book. Adopt the *We-Do* strategy in the demonstration.
Note: *We-Do* strategy involves the teacher engaging students in the demonstration through probing and prompting them for responses.

Ask: How can we express $(a+b)^3$ as a product of two factors, then write $(a+b)(a+b)^2$? Can $(a+b)^3$ also be expressed as $(a+b)^2(a+b)$? How can we perform the expansion?

TWM1 (TWM.03 Conjecturing)
Students will demonstrate that they are **conjecturing** when they are asked to predict the sum of terms in the expansion of $(a+b)^3$.

Guide students to carry out the expansion.
Step 1: Substitute $(a+b)^2$ by $(a^2+2ab+b^2)$
Step 2: Demonstrate the expansion using Distributive Law (as shown in **Take Note**)

Pause and Address: After expanding $(a+b)^3$, remind students that the sum is not a^3+b^3.

Demonstrate another method of expansion using the Frame (or Box) Method as shown below:

(i) Draw a 3 × 4 grid on the board.
Ask: Do you know why we use a 3 by 4 grid in this example?

(ii) Place the binomial factor in the first column and the trinomial factor in the top row

$\times$	$\boldsymbol{a^2}$	$\boldsymbol{+2ab}$	$\boldsymbol{+b^2}$
$\boldsymbol{a}$			
$\boldsymbol{+b}$			

(iii) Multiply *a* with each term in the trinomial as follows:

$\times$	$\boldsymbol{a^2}$	$\boldsymbol{+2ab}$	$\boldsymbol{+b^2}$
$\boldsymbol{a}$	$\boldsymbol{a^3}$	$\boldsymbol{2a^2b}$	$\boldsymbol{ab^2}$
$\boldsymbol{+b}$			

(iv) Multiply *b* with each term in the trinomial as follows:

$\times$	$\boldsymbol{a^2}$	$\boldsymbol{+2ab}$	$\boldsymbol{+b^2}$
$\boldsymbol{a}$	$\boldsymbol{a^3}$	$\boldsymbol{2a^2b}$	$\boldsymbol{ab^2}$
$\boldsymbol{+b}$	$\boldsymbol{a^2b}$	$\boldsymbol{2ab^2}$	$\boldsymbol{b^3}$

(v) Sum up the six terms in the coloured boxes.

$\times$	$\boldsymbol{a^2}$	$\boldsymbol{+2ab}$	$\boldsymbol{+b^2}$
$\boldsymbol{a}$	$\boldsymbol{a^3}$	$\boldsymbol{2a^2b}$	$\boldsymbol{ab^2}$
$\boldsymbol{+b}$	$\boldsymbol{a^2b}$	$\boldsymbol{2ab^2}$	$\boldsymbol{b^3}$

Focus students' attention on the two pairs of like terms in the light grey and dark grey boxes respectively. Sum up the two terms in the light grey boxes to give $3a^2b$ and sum up the two terms in the dark grey boxes to give $3ab^2$.

Show students by summing up all the terms in the coloured boxes, the expression becomes

$$a^3+3a^2b+3ab^2+b^3$$

Invite students to compare both methods of expansion and share their preferred method. Ask them why they prefer that method over the other, and identify and list out the possible advantages and disadvantages of both methods.

TWM2 (TWM.07 Critiquing)
Students will demonstrate that they are **critiquing** when they compare and contrast the two methods of expansion and list the possible advantages and disadvantages of both methods.

(b) Ask students to attempt the expansion of $(a+b)^4$ on their own by filling in the boxes.

$$
\begin{aligned}
(a+b)^4 &= (a+b)(a+b)^{[3]} \\
&= (a+b)\,(a^3+3a^2b+2ab^2+b^3) \\
&= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4 \\
&= a^4+4a^3b+[6]a^2b^2+[4ab^3]+[b^4]
\end{aligned}
$$

Ask: How do you know your expansion is correct?

TWM3 (TWM.04 Convincing)
Students will demonstrate that they are **convincing** when they are able to explain the veracity of their solution using other methods.

Encourage students to consider expanding $(a+b)^4$ using the Frame (or Box) Method to check their answer.

> **Pause and Think**
> Ask students to expand $(a+b)^4$ by using $(a+b)^2$.
> Consider $(a+b)^4 = (a+b)^2(a+b)^2$.

Ask students to compare the two methods: $(a+b)^4 = (a+b)(a+b)^3$ and $(a+b)^4 = (a+b)^2(a+b)^2$, and discuss the efficiency of both methods.

TWM4 (TWM.07 Critiquing)
Students will demonstrate that they are **critiquing** when they compare and contrast the two methods of expansion and list the possible advantages and disadvantages of both methods.

(c) Ask students to expand $(a+b)^5$. Repeat the steps as in **(b)** using $(a+b)^5 = (a+b)^3(a+b)^2$.

(d) The expansion of $(a+b)^n$ for $n =$ 1, 2, 3, 4 and 5 is a sum of terms. Have students extract the information from **(a)** to **(c)** above and record them in the table.

Binomial expression	***n***	**Expansion in descending powers of *a***	**Number of terms in the expansion**	**Coefficient of the second term**
$(a+b)$	1	$a+b$	2	1
$(a+b)^2$	2	$a^2+2ab+b^2$	3	2
$(a+b)^3$	3	$a^3+3a^2b+3ab^2+b^3$	4	3
$(a+b)^4$	4	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$	5	4
$(a+b)^5$	5	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$	6	5

Guide students to fill in the table with the following questions:

- *Compare entries in columns 2 and 4. What connection can you establish between the two columns?*
- If the value of $n = 6$, how many terms are there in the expansion?
- *Why does the last term not contain the variable a?*
- *Follow the decreasing powers of n in a as the terms go from left to right. What is the value of a^0?* (Adopt the same approach for b. Highlight that the powers of b is increasing from left to right and goes from 0 to n.)

If students are unable to see the connection, extend the pattern by asking them to determine the number of terms when $n = 6$ and 7, then skip a few values to $n = 10$.
Skip more values to $n = 20$ until students can detect the pattern.

TWM6 (TWM.01 Specialising)
Students will demonstrate that they are **specialising** when they consider specific cases in order to make sense and detect a pattern.

2. Pascal's Triangle

- Go through the Pascal's Triangle. Prompt students to look at the coefficient of the 2nd term in the expansion for $n = 2$ to 5.
  Ask: What do you notice about the coefficient and the power?

- Guide students to observe the pattern by writing the coefficient of the terms in each expansion for $n = 2$ to 5 below:

n	Coefficients of the terms in the expansion $(a+b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Ask: What pattern do you notice?
A possible answer that students may give is that the numbers are 'repeating'.
Clarify their thinking by explaining that the correct word should be 'symmetrical'.

References:

1	Patterns in Pascal's Triangle	• https://www.mathsisfun.com/pascals-triangle.html
2	History on Pascal's Triangle	• Kennedy, E. (1966). Omar Khayyam. The Mathematics Teacher 1958. National Council of Teachers of Mathematics. pp. 140–142. • Edwards, A.W.F. (2013). The Arithmetical Triangle. In R. Wilson & J. J. Watkins (Eds.), *Combinatorics: Ancient and Modern* (pp. 166–180). Oxford: Oxford University Press • Sidoli, N., van Brummelen, G., & Brummelen, V. G. (2013). *From Alexandria, Through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J.L. Berggren* (1st ed.). Springer. • Foster, T., & Peters, V. L. (2014). Nilakantha's Footprints in Pascal's Triangle. *Mathematics Teacher, 108*(1), 247 – 255.

Key Results (p. 5)

Guide students to make five observations about the expansion of $(a+b)^n$.

TWM5 (TWM.02 Generalising)

Students are **generalising** when they study the entries in the table and recognise underlying patterns by identifying what remains the same and what changes.

- **Observation 1:**
  - There are $(n+1)$ terms in the expansion.

- **Observation 2:**
  - Focus students' attention on the powers of a when $n = 2$, 3, 4 and 5 case. For the last term in the expansion b^n, it may not be obvious to most students that the power of a is zero. This is where the teacher can ask students to explain why they think this last term does not contain a.
  - *Ask: If we were to follow the pattern of decreasing powers of n, what would the power of a be?* What is the value of a^0?
  - Teacher should then highlight that the powers of a are said to be **decreasing**.
  - To fill in the blank for the second sentence, adopt a similar approach as above. Conclude that the powers of b are said to be **increasing**.
  - The power of a starts with n and decreases to 0.
    The power of b starts with 0 and increases to n .

- **Observation 3:**
  - Focus students' attention on the 2nd term in each expansion for $n = 2$ to 5, then compare the respective coefficients of the 2nd term with the values of n in Column 2.
  - The coefficient of the second term is always n.

- **Observation 4:**
  - Write the coefficients of the terms in each expansion for n = 2 to 5 on the board as follows:

n	
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

  - Ask students what they notice about the coefficients in each expansion. They may not think of the right word to describe it. A possible student response could be 'the coefficients are **repeating**'. The word is incorrect because repeating can occur in unit blocks, for e.g., 1 3 1 3 (formed by a 2-digit unit block '1 3') or 1 4 6 1 4 6 (by a 3-digit unit block '1 4 6)'.
    The expected word for the blank should be **symmetrical**.

- **Observation 5:**
  - Have students to consider every term in each expansion when $n = 2$ to 5.
    Demonstrate that for $n = 2$, the powers of a and b for a^2 are 2 and 0 respectively.
    So, the sum of the powers is 2.
  - For $2ab$, the powers of a and b are 1 and 1 respectively. So, the sum of the powers is also 2.
    For b^2, the powers of a and b are 0 and 2 respectively. Again, the sum of the powers is 2.

- In pairs, ask students to check the sum of powers for each term from $n = 3$ to 5.
- Conclude that the sum of the powers of *a* and *b* in each term is always $\underline{n}$.

Pause and Think
Prompt students to think of the expansion of $(a + b)^n$ when $n = 0$.
Recall with students what a^0 is for $a > 0$.
Ask: Since $5^0 = 1$, what is the value of $(1 + 4)^0$? or $(3 + 2)^0$? [Answer: 1]

TWM6 (TWM.01 Specialising)
Students will demonstrate that they are **specialising** when they relate back to any integer raised to power 0.

Guide students to conclude their observations of the Pascal's Triangle.
In the Pascal's Triangle,
- each row starts and ends with 1.
- the remaining numbers in each row can be obtained from the previous row by adding together the two numbers in the previous row which are closest to it.

Wrap-up
Review the lesson by summarising the key points. By the end of the lesson, students should understand how to:
- write the expansion of $(a + b)^n$
- use Pascal's Triangle to expand $(a + b)^n$

LESSONS 2 AND 3

Warm-up
Briefly recap what students have learnt about binomial expansion and the Pascal's Triangle from the earlier lesson. Explain that they will need to apply the Key Results from the previous lesson to go through the worked examples in the lesson.

Main Lesson Content
Go through **Worked Examples 1 to 6**. At relevant junctures after going through the worked examples, have students attempt **Pause and Try 1 to 6**.

Worked Example 1 and Pause and Try 1 (p. 6)
- Students will learn to expand binomial with sum of two variables using the Pascal's Triangle.
- Guide students to apply the observations made in Key Results in the previous lesson to solve the problem.
- *Ask: What are the five observations you made about $(a + b)^n$? How can you apply the observations to solve the problem?*
[Answer: There are $(n + 1)$ terms in the expansion, so there are 7 terms in the expansion, etc.]

Worked Example 2 and Pause and Try 2 (p. 7)
- Students will learn to expand binomial with difference of two variables using the Pascal's Triangle.
- Remind students to be careful when simplifying $(-b)^n$ for $n = 0$ to 6.
- ***Pause and Address:*** Have students pay attention to the common error highlighted in **Take Note!**

Worked Example 3 and Pause and Try 3 (p. 8)

- Students will learn to expand binomial with sum of a variable and an integer using the Pascal's Triangle.
- After completing the expansion of $(a + 2)^5$, have students pay attention to the sum of terms.
- ***Pause and Address:*** Point out to students that the sum is a polynomial of degree 5 in a. Get students to identify the polynomial in **Try 3**.

Worked Example 4 and Pause and Try 4 (p. 9)

- Students will learn to expand the product of a binomial and a higher power of another binomial using the Pascal's Triangle.
- Explain that the expansion of the product, the term with the lowest power of k is obtained from multiplying the constant terms. This is because the power of k is zero. Therefore, the first three terms in the expansion of $(1 + k)(1 - 2k)^4$, in ascending powers of k, are the terms up to k^2.
- To obtain the constant term and the terms in k and k^2, we just need to multiply 1 in $(1 + k)$ with the first three terms in the expansion of $(1 - 2k)^4$ and k in $(1 + k)$ with the first two terms in the expansion of $(1 - 2k)^4$.

Worked Example 5 and Pause and Try 5 (p. 10)

- Students will learn to find the coefficient of a term in the expansion of the product of a binomial and a higher power of another binomial.
- To determine the term in k^3 in the expansion of the product, we multiply 1 with $-32k^3$ and $2k$ with $24k^2$.

Worked Example 6 and Pause and Try 6 (p. 10)

- Students will learn to estimate powers of a real number.
- ***Pause and Address:*** Go through the second **Take Note!** with students, explaining
  - how the value of *x* = 0.01 is chosen for substitution into $(1 - 2x)^7$,
  - how the substitution into the sum works.

Wrap-up

Review the lesson by summarising the key points. By the end of the lesson, students should understand:

- steps and procedures in solving the problems involving expansion of sum or difference of binomial terms using the Pascal's Triangle
- expansion of binomials in ascending and descending order of powers of *a* or *b*.
- expansion of terms of the product of a binomial and higher power of another binomial
- estimates of a real number using binomial expansion

Assign **Exercise 12.1** for students to do as practice for consolidation of concepts and skills. For further practice, assign **Further Exercise 12.1** in the Workbook to students.

Lesson Plans for Section 12.2 on
Expansion of $(a + b)^n$ using Binomial Theorem
are not included in this submission.

Wrap-up

Review the lesson by summarising the key points. By the end of the lesson, students should be able to:

- expand using Binomial Theorem
- apply the general term formula to solve problems
- estimate powers of a real number

Assign **Exercise 12.2** for students to do as practice for consolidation of concepts and skills. For further practice, assign **Further Exercise 12.2** in the Workbook to students.

Direct students to the list of learning outcomes under **Review** and have them put a tick in the boxes for all the learning outcomes that they have achieved.

Assign **Review Questions** as practice for students to consolidate and reinforce the concepts and skills covered in the chapter.

Go through the **Chapter Opener Revisit** in the Workbook. Students should be able to apply what they have learned in this chapter to solve the problem presented in the Chapter Opener.

- Go through a similar process detailed in **Worked Example 9** on page 16 to solve the problem in the Chapter Opener.
- The exact amount of money can be found by $\$500 \times 1.01^{10}$. Point out that finding the value of numbers such as 1.01^{10} is tedious. Have students use the Binomial Theorem to estimate the value of such numbers.
- By using the expansion of $(1 + 0.01)^{10}$, have students estimate the amount of savings in the bank at the end of 10 years.

  $$\begin{aligned}(1 + 0.01)^{10} &= 1 + 10(0.01) + 45(0.01)^2 + 120(0.01)^3 + \cdots \\ &= 1.10462\end{aligned}$$
- Explain that the estimated value obtained from the first four terms of the binomial expansion of $(1 + 0.01)^{10}$ is already sufficient to give students an accurate estimation of the amount of money in the bank at the end of 10 years.
- To challenge the more abled students, you may wish to extent the question in the Chapter Opener by posing the question below:
- *Suppose that you put an initial amount of money for 10 years at a given compound interest rate. Using the first three terms of a binomial expansion to approximate the final amount y in the bank, you will find that the error is \$0.01. Does this mean that the relative error is 1%? If not, find the condition on y such that the relative error is 1%.*

To conclude the chapter, go through **Fun with Maths!** in the Workbook. This task encourages students to expand binomials using an alternative method.

- Partition the whiteboard into two parts. The left part is for writing out the steps given in the textbook. The right part is for illustrating the example of expanding $(a+b)^6$.
- Introduce the steps, one at a time, using the expansion of $(a+b)^6$.
- Present Step 1 as follows:

Steps for expanding $(a+b)^n$	Illustration using $(a+b)^6$
The first term is a^n.	The first term is a^6

Then present Step 2:

Steps for expanding $(a+b)^n$	Illustration using $(a+b)^6$
The coefficient of the second term is *n* and its variable component is $a^{n-1}b$.	The coefficient of the second term is 6. The variable component is a^5b. So, the second term is $6a^5b$.

- As you state the variable component, remind students that the power of *a* is 1 less than the first term and the power of *b* is 1 more. Remind students to check that the sum of powers is 6.
- Then go through Steps 3 and 4. In Step 3, take time to explain the procedures slowly because this is the important step of the algorithm.